Resonance Characteristics of Whispering-Gallery Modes in an Elliptic Dielectric Disk Resonator

Yoshinori Kogami, Yoshiro Tomabechi, and Kazuhito Matsumura

***Abstract*—This paper presents a novel calculation method for resonance characteristics of an elliptic dielectric disk resonator. In the elliptic resonator, electromagnetic energy propagates as a whispering gallery (WG) mode along the edge region of the dielectric disk. In the novel analysis, a local propagation constant of the WG mode at each point of the elliptic disk edge is represented by a propagation constant of a WG mode in a circular disk whose radius of curvature, material constants, and thickness are equal to those of the elliptic disk. Therefore, we can easily calculate the resonance characteristics of the elliptic dielectric disk resonator by applying a conventional technique for the circular dielectric disk. The calculated results of the resonant frequencies and field distribution are well confirmed by experiments.**

I. Introduction

Whispering gallery (WG) mode dielectric resonators that have very large dimensions compared to their handling wavelength have been proposed for a high-performance device in millimeter-wave integrated circuits [1]–[7]. Many researchers have investigated the characteristics of the WG modes in circular dielectric disk resonators and have shown their merits, such as high quality factor and adaptability for circuit integration [1]–[7]. An elliptic dielectric disk resonator inherits these merits from the circular disk and will have an apparent advantage with its field distribution. The field strength on the circumference of the elliptic disk varies with position; that is, the external field around a major axis is stronger than that around a minor axis [8]. This feature of the elliptic disk resonator is very attractive because it has been an important subject to obtain an adequate strong coupling in practical use of the WG mode dielectric resonator, especially of high-permittivity material. This type of the elliptic disk resonator, however, has never been analyzed rigorously.

In this paper, an approximate approach is presented to calculate the resonant frequency and the field distribution of the elliptic disk resonator. The WG mode in the elliptic dielectric disk propagates, changing its propagation constant, along the circumference of the disk. The propagation constant and field distribution strongly depend on its radius of curvature [7]. If we restrict our consideration to a small region near a point on the circumference, we can consider that the local propagation characteristics and field pattern of the WG mode are almost similar to those of a circular disk. This circular disk is inscribed to the ellipse at the point and has the same curvature and the same thickness as the elliptic disk. The propagation constants of the WG mode in the elliptic disk can be determined from the local propagation constants and the local fields, which are calculated for the inscribed circular disks at the every point on the circumference. Therefore, we can obtain the resonant frequency and field distribution of the WG mode elliptic disk resonator. In the following, an analytical procedure based on the above point of view is described in detail. Furthermore, the calculated results for an X band model are shown and are compared with measured data.

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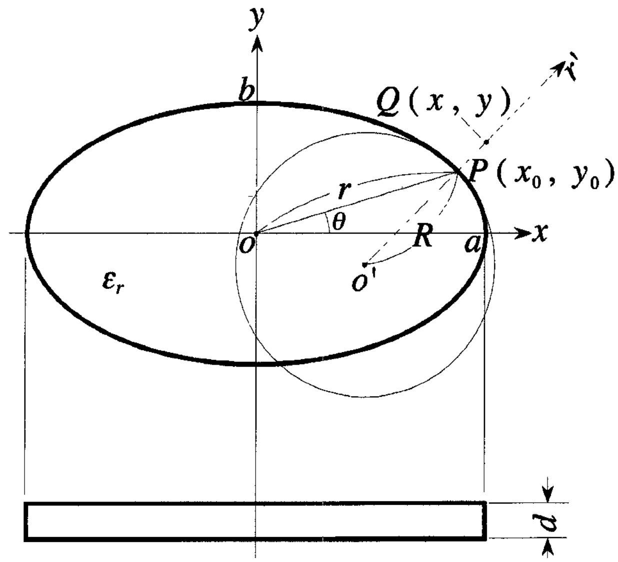


Fig. 1. Elliptic disk and inscribed circular disk.

II. Analysis

A. Resonant Frequency of the WG Mode

Fig. 1 shows an elliptic dielectric disk that has the major axis a, minor axis b, thickness d, and relative permittivity ϵ_r. The lengths of a and b are tens to hundreds times longer than the wavelength, while d is of the same order as the wavelength.

To calculate a local propagation constant $\gamma(\theta) = \alpha(\theta) + j\beta(\theta)$ of the WG mode, we introduce an inscribed circular disk at a point $P(r, \theta)$ on the circumference of the ellipse, as shown in Fig. 1. The inscribed disk has a radius R, which is the same as the radius of curvature of the ellipse at the point P; that is

$$R = \frac{(a^4 y_0^2 + b^4 x_0^2)^{3/2}}{a^2 b^2 (a^2 y_0^2 + b^2 x_0^2)} \tag{1}$$

where

$$x_0 = r\cos\theta = \frac{ab\cos\theta}{\sqrt{a^2\sin\theta^2 + b^2\cos\theta^2}},$$

$$y_0 = r\sin\theta = \frac{ab\sin\theta}{\sqrt{a^2\sin\theta^2 + b^2\cos\theta^2}}. \tag{2}$$

When P is on the major axis, R has a minimum value of $R_{\min}$. When P is on the minor axis, R reaches its maximum value of $R_{\max}$. Otherwise the ratio of $R_{\max}$ to $R_{\min}$ increases with increasing eccentricity of the ellipse.

The local propagation constant of the WG mode for the inscribed circular disk can be obtained by solving the following equations [6], [7]

$$\left.\begin{aligned} &\tan\left(k_z \tfrac{d}{2}\right) = \epsilon_r \tfrac{\gamma_z}{k_z} \\ &\tfrac{1}{\rho} J_\nu(\sigma R) H_\nu^{(2)'}(\rho R) = \tfrac{\epsilon_r}{\sigma} J_\nu'(\sigma R) H_\nu^{(2)}(\rho R) \end{aligned}\right\} \tag{3}$$

where

$$\nu = (\beta - j\alpha)R$$

$$k_z^2 + \gamma_z^2 = \epsilon_r k_0^2, \quad k_0\text{: wave number in free space}$$

$$\rho = \sqrt{\epsilon_r k_0^2 - k_z^2}, \quad \sigma = \sqrt{k_0^2 - k_z^2}.$$

In this analysis, we consider that the propagation constant for the inscribed circular disk obtained from (3) is equal to the local propagation constant for the WG mode in the elliptic disk at the point P. Moving the point P along the circumference of the ellipse, we can obtain the local propagation constants on the elliptic disk continuously.

A total phase shift of the propagating WG mode, which makes one revolution around the elliptic disk, is obtained by an integration of the local phase constant $\beta(\theta)$ along the circumference. The WG mode resonance arises at a frequency where the total phase shift equals to an integer multiple of 2 π; that is

$$\int_0^{2\pi} \beta(\theta) l(\theta)\, d\theta = 2N\pi \tag{4}$$

where, $l(\theta)d\theta$ is the differential length on the disk edge and integer N is an azimuthal resonance index.

B. Field Distribution of the WG Mode

The field distribution is also calculated by using the inscribed circular disks. At first, we calculate the field distribution on a line which is drawn from the center of an inscribed circular disk O' through the inscribed point $P(x_0, y_0)$ as shown in Fig. 1. For example, the distribution of z component of the electric field along the line $O'P$ is calculated from

$$E_z(\theta, r') = \begin{cases} A(\theta) J_\nu(\sigma r') \cos(k_z z) e^{-j\nu\theta} & \cdots r' \le R \\ B(\theta) H_\nu^{(2)}(\sigma r') \cos(k_z z) e^{-j\nu\theta} & \cdots r' > R \end{cases} \tag{5}$$

where r' is a distance between the center of the inscribed disk and a point $Q(x, y)$ on the line and is related with x and y as

$$x = x_0 + (r' - R)\sin\{\tan^{-1}(a^2 y_0 / b^2 x_0)\}$$

$$y = y_0 + (r' - R)\cos\{\tan^{-1}(a^2 y_0 / b^2 x_0)\}. \tag{6}$$

Moving the point P, we repeat this calculation so that we obtain the field intensity on the point $Q(x, y)$, which covers the region of interest near the edge of elliptic disk. In this calculation, we assume that the radiation and dielectric losses are all represented by the local attenuation constant α and the amplitude decreases with only the quantity of α; that is

$$A(\theta) = A_0 e^{-\int_0^\theta \alpha\, dl}, \quad dl = ab(2a^2\cos\theta - 2b^2\sin\theta)^{-\frac{3}{2}}\, d\theta \tag{7}$$

where A_0 is an initial value at $\theta = 0$. $B(\theta)$ can be calculated from $A(\theta)$ and the field continuity at boundary $r' = R$. By using the estimated $A(\theta)$ and $B(\theta)$, we can link the field amplitudes on the point Q and obtain the total field distribution of the elliptic disk resonator.

C. Numerical Results

The present method is employed to calculate the resonant frequencies and the field distribution of an elliptic dielectric disk resonator. The dimensions and dielectric constants of the resonator are shown in Table I.

Generally, WG modes are classified in quasi-TM families and quasi-TE families. In this calculation, we are interested in the quasi-TM families only. Fig. 2 shows calculated results of the local phase constant $\beta(\theta)$ and the local attenuation constant $\alpha(\theta)$. The WG mode waves propagate fast around the minor axis and slow down around the major axis, where the main part of the radiation loss arises. As described above, $\beta(\theta)$ is used in (3) to calculate the resonant frequency and $\alpha(\theta)$ is used in (7) to determine the amplitude of the local field. Table II shows the resonant frequencies calculated numerically for the some WG modes in the elliptic disk. Frequency separation between adjacent resonant modes is about 195 MHz.

Also, the electric field distribution for a WG mode of $N = 43$ in the elliptic disk is calculated by presented method and is expressed by an equi-amplitude contour map in Fig. 3. It is found that the WG

TABLE I
DIMENSION OF THE ELLIPTICAL DIELECTRIC DISK

Major axis : a (mm)	200
Minor axis : b (mm)	140
Thickness : d (mm)	20
Relative permittivity ϵ_r	2.01

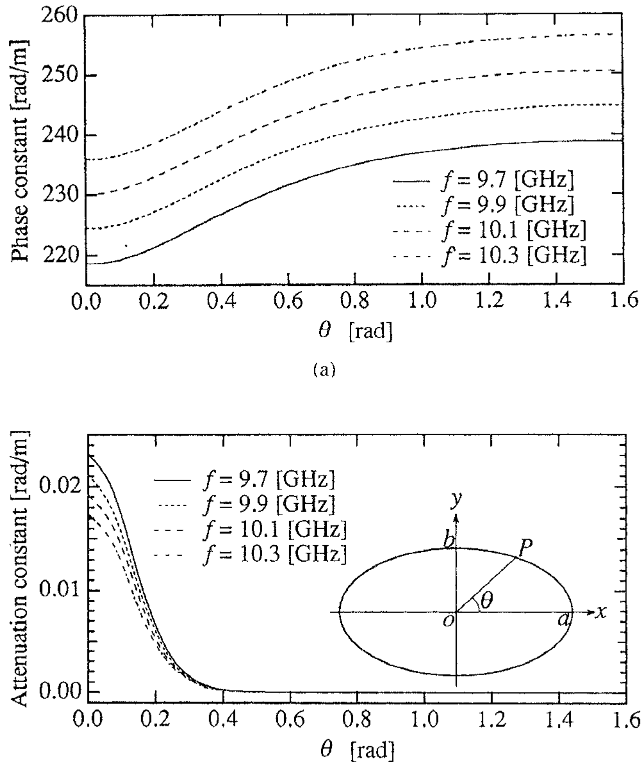


Fig. 2. Calculated results of local propagation constant. ($\epsilon_r = 2.01$, $a = 0.10$ m, $b = 0.20$ m, $d = 0.02$ m). (a) Phase constant. (b) Attenuation constant.

TABLE II
CALCULATED AND MEASURED RESONANT FREQUENCIES

Resonance order		39	40	41	42	43
Resonant frequency	Calculated (GHz)	9.541	9.735	9.928	10.121	10.315
	Measured (GHz)	9.563	9.758	9.954	10.149	10.348

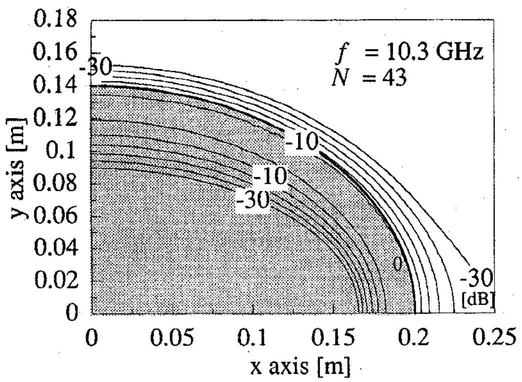
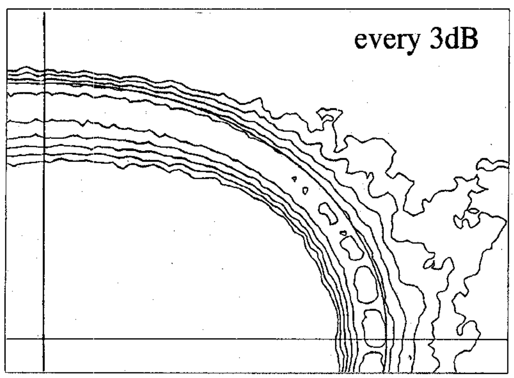
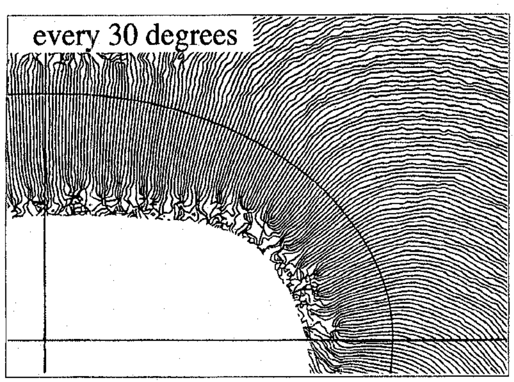


Fig. 3. Calculated results of field distribution. ($\epsilon_r = 2.01$, $a = 0.10$ m, $b = 0.20$ m, and $d = 0.02$ m).



(a)



(b)

Fig. 4. Measured results of field distribution. ($\epsilon_r = 2.01$, $a = 0.10$ m, $b = 0.20$ m, $d = 0.02$ m, $f = 10.348$ GHz). (a) Field intensity. (b) Phase.

mode field is concentrated near the edge of the disk, but near the major axis more of the field escapes into the outer region.

D. Experimental Results

We measured the resonant frequencies and the electric field distribution of an elliptic dielectric disk, which is made of tetra fluoro ethylene (Teflon) whose dimensions are shown in Table I. The measured results of resonant frequencies are shown in Table II. The deviation between the calculated and measured resonant frequencies is only about 30 MHz. Fig. 4(a) is an equi-amplitude contour map of the measured electric field. Fig. 4(b) is an equi-phase contour map of the measured field in which the contours are drawn 30° apart. By counting the contour lines, we can find that the resonance order N is 43, which is the same as one calculated above. The space between the equi-phase lines near the major axis region is slightly narrower than that near the minor axis region. This tendency agrees with the calculated results for the local phase constant shown in Fig. 2.

III. Conclusion

A calculation method of resonant frequencies and field distributions is presented for WG modes in an elliptic dielectric disk resonator. Phase and attenuation constants of the WG mode that propagates along the circumference of the elliptic disk are obtained from those of the inscribed circular disks. The resonant frequency is determined by the resonance condition where the circuit integration of the local phase constants along the circumference equals to $2N\pi$. The field distribution is also obtained from the local field amplitudes calculated for the inscribed disks.

Calculated results of resonant frequencies and electric field distribution for an X band model are confirmed experimentally. The field to penetrate into outer region near the major axis of the elliptic disk is observed in the calculated and measured results of field distribution.

The presented calculation method can be used to estimate easily the characteristics of the elliptic dielectric disk resonator and to determine the location of external circuits for adequate coupling. In the future, the elliptic dielectric disk resonator, which will be made from low-loss and high-permittivity dielectric material, will be applied in high-performance components, such as filters and power combiners, on millimeter-wave integrated circuits.

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